THEORY OF THE "EQUIVALENT SLOPE"

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Potential insolation as a variable dependent upon the orientation of actual or theoretical plane surfaces is of interest to architects, hydrologists, and land managers. Integrated daily totals of the extraterrestrial insolation available on horizontal surfaces at any latitude have been published by The Smithsonian Institution [3]. Conversion of these values to account for the inclination of any plane is a function of the theory in question.

The "equivalent slope" concept is derived from the fact that every inclined surface on the face of a sphere is parallel to some horizontal surface whose location is mathematically defined. The determination of the location of this equivalent slope in terms of increments of latitude and longitude requires the solution of a terrestrial spherical triangle. In figure 1, for example, we may choose triangle ABC where A represents the North Pole and B and C are the points of direct concern.

There is a serious disorder in the works of Kimball [2] and in Bates and Henry [1] with respect to the method of computing geographic locations of the equivalent slope. The theory of the equivalent slope was stated by Kimball [2] as follows: "In the case of a slope facing α' degrees in azimuth the angle of incidence of the solar rays will be the same as on a horizontal surface at a point on a great circle passing through the slope at right angles to it and as many degrees removed as the angle of the slope. We may locate this point in latitude and longitude by the solution of the . . . spherical triangle . . . ." However, the equations used by Kimball [2] and Bates and Henry [1] neglect the "great circle" requirement.

Kimball's equations are

$$\tan \Delta L = (\cos h)(\tan k) \tag{1}$$

and

$$\sin \Delta T = (\sin h)(\sin k) \tag{2}$$

where ΔL is the difference in latitude between slope of concern and equivalent slope; ΔT is the difference in longitude; h is the azimuth of the slope from north; and k is the slope inclination.

These equations yield the correct solution only when the face of the inclined surface is bisected orthogonally by the equator or a meridian; i.e., the slope is either north- or south-facing, or is an east- or west-facing incline at the equator. We see, for example, that an east- or west-facing slope of $k°$ located at any latitude greater than 0° is *not*

parallel to a horizontal surface that lies simply $k°$ east or west of itself, as equation (2) would indicate. It is upon this consideration that the necessity for Kimball's "great circle" stipulation becomes apparent. In order to obtain the location of the equivalent slope it is necessary to recognize—even for east- and west-facing facets—a change in latitude as well as in longitude. Thus a 30° east-facing facet at 40° N. latitude is found to be parallel to a horizontal surface 6° 11' south and 37° 00' east of itself, rather than 30° due east as the combined equations specify.

The correct difference in longitude between the location of a given slope and that of an equivalent horizontal surface (between B and C in fig. 1) is given by

$$\Delta T = \tan^{-1}\left(\frac{\sin h \cdot \sin k}{\cos k \cdot \cos \theta - \cos h \cdot \sin k \cdot \sin \theta}\right) \tag{3}$$
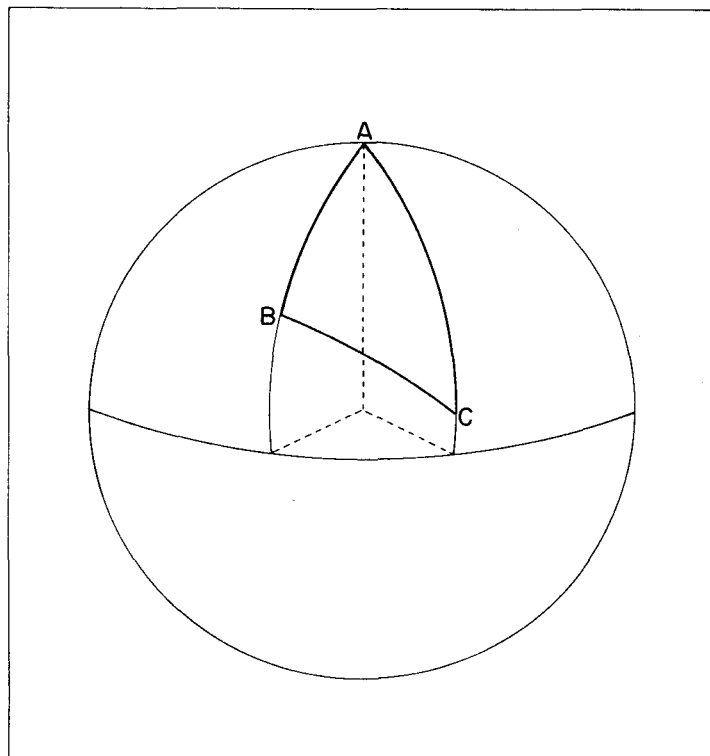

FIGURE 1.—The terrestrial spherical triangle, $\triangle ABC$.

where θ is latitude (N+, S−). The difference in latitude between points B and C of figure 1 is given by

$$\Delta L = \text{Colatitude } B - \text{Colatitude } C \qquad (4)$$

where Colatitude B is known and Colatitude C is given by

$$\text{Colatitude } C = \sin^{-1}(\sin h \cdot \sin k / \sin \Delta T) \qquad (5)$$

or independently

$$\text{Colatitude } C = \cos^{-1}(\sin k \cdot \cos h \cdot \cos \theta + \cos k \cdot \sin \theta) \qquad (6)$$

We notice at once the inclusion of latitude as a variable, the lack of which might have caused us to suspect equations (1) and (2) upon inspection.

The seriousness of the discrepancies in results obtained by equations (1) and (2) and by (3) and (4) is illustrated by specific examples from Bates and Henry [1] in which Kimball's equations were used (table 1).

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TABLE 1.—*Corrected coordinates of equivalent slopes compared with coordinates derived by Kimball*

Given Slope			Equivalent Slope		
θ	h	k		ΔL	ΔT
				° ′	° ′
37°46′	336°	31°20′	(K)[1]	29 05	12 13
			(C)	26 13	28 50
			(E)	2 52	−15 37
37°46′	124°	34°20′	(K)	−20 54	−27 53
			(C)	−22 50	−28 56
			(E)	1 56	1 03
37°46′	24°	37°30′	(K)	35 02	−14 20
			(C)	30 04	−40 50
			(E)	4 58	26 30
37°46′	135°	30°00′	(K)	−22 12	−20 42
			(C)	−23 20	−21 18
			(E)	1 08	0 36

[1] (K), Computed by Kimball's equations; (C), by corrected equations; (E), difference.

REFERENCES

1. C. G. Bates and A. J. Henry, "Forest and Streamflow Experiment at Wagon Wheel Gap, Colorado," *Monthly Weather Review Supplement* No. 30, U.S. Weather Bureau, Washington, D.C., 1928, 79 pp.
2. H. H. Kimball, "Variations in the Total and Luminous Solar Radiation with Geographical Position in the United States," *Monthly Weather Review*, vol. 47, No. 11, Nov. 1919, pp. 769–793.
3. The Smithsonian Institution, *Smithsonian Meteorological Tables*, 6th rev. ed., Washington, D.C., 1951 (pp. 417–418).